

THE GROEBNER BASIS OF A POLYNOMIAL SYSTEM

CHRISTIAN VALQUI AND MARCO SOLORZANO

ABSTRACT. We compute the Groebner basis of a system of polynomial equations related to the Jacobian conjecture using a recursive formula for the Catalan numbers.

1 Introduction

In this paper K is a characteristic zero field and $K[y]((x^{-1}))$ is the algebra of Laurent series in x^{-1} with coefficients in $K[y]$. In a recent article the following theorem was proved [2, Theorem 1.9].

Theorem 1.1. *The Jacobian conjecture in dimension two is false if and only if there exist*

- $P, Q \in K[x, y]$ and $C, F \in K[y]((x^{-1}))$,
- $n, m \in \mathbb{N}$ such that $n \nmid m$ and $m \nmid n$,
- $\lambda_i \in K$ ($i = 0, \dots, m+n-2$) with $\lambda_0 = 1$,

such that

- C has the form

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \dots \quad \text{with each } C_{-i} \in K[y],$$

- $gr(C) = 1$ and $gr(F) = 2 - n$, where gr is the total degree,
- $F_+ = x^{1-n}y$, where F_+ is the term of maximal degree in x of F ,
- $C^n = P$ and $Q = \sum_{i=0}^{m+n-2} \lambda_i C^{m-i} + F$.

Furthermore, under these conditions (P, Q) is a counterexample to the Jacobian conjecture.

Motivated by this result, the authors consider the following slightly more general situation. Let D be a K -algebra (in Theorem 1.1 we take $D = K[y]$), n, m positive integers such that $n \nmid m$ and $m \nmid n$, $(\lambda_i)_{1 \leq i \leq m+n-2}$ a family of elements in K with $\lambda_0 = 1$ and $F_{1-n} \in D$ (in Theorem 1.1 we take $F_{1-n} = y$). A Laurent series in x^{-1} of the form

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \dots \quad \text{with } C_{-i} \in D,$$

is a solution of the system $S(n, m, (\lambda_i), F_{1-n})$, if there exist $P, Q \in D[x]$ and $F \in D[[x^{-1}]]$, such that

$$F = F_{1-n}x^{1-n} + F_{-n}x^{-n} + F_{-1-n}x^{-1-n} + \dots, \quad (1.1)$$

$$P = C^n \quad \text{and} \quad Q = \sum_{i=0}^{m+n-2} \lambda_i C^{m-i} + F. \quad (1.2)$$

2010 *Mathematics Subject Classification.* Primary 14R15; Secondary 13F20, 11B99.

Key words and phrases. Jacobian, Groebner basis, Catalan numbers.

Christian Valqui was supported by PUCP-DGI-2013-3036.

For example, if $n = 2$, then

$$\begin{aligned} P(\mathbf{x}) = C^2 = & \mathbf{x}^2 + 2C_{-1} + 2C_{-2} \mathbf{x}^{-1} + (C_{-1}^2 + 2C_{-3}) \mathbf{x}^{-2} + (2C_{-1}C_{-2} + 2C_{-4}) \mathbf{x}^{-3} \\ & + (C_{-2}^2 + 2C_{-1}C_{-3} + 2C_{-5}) \mathbf{x}^{-4} + (2C_{-2}C_{-3} + 2C_{-1}C_{-4} + 2C_{-6}) \mathbf{x}^{-5} + \dots, \end{aligned}$$

and the condition $C^2 \in K[x]$ translates into the following conditions on C_{-k} :

$$\begin{aligned} 0 &= (C^2)_{-1} = 2C_{-2}, \\ 0 &= (C^2)_{-2} = C_{-1}^2 + 2C_{-3}, \\ 0 &= (C^2)_{-3} = 2C_{-1}C_{-2} + 2C_{-4}, \\ 0 &= (C^2)_{-4} = C_{-2}^2 + 2C_{-1}C_{-3} + 2C_{-5}, \\ 0 &= (C^2)_{-5} = 2C_{-2}C_{-3} + 2C_{-1}C_{-4} + 2C_{-6}, \\ 0 &= (C^2)_{-6} = C_{-3}^2 + 2C_{-2}C_{-4} + 2C_{-1}C_{-5} + 2C_{-7}, \\ 0 &= (C^2)_{-7} = 2C_{-3}C_{-4} + 2C_{-2}C_{-5} + 2C_{-1}C_{-6} + 2C_{-8}, \\ 0 &= (C^2)_{-8} = C_{-4}^2 + 2C_{-3}C_{-5} + 2C_{-2}C_{-6} + 2C_{-1}C_{-7} + 2C_{-9}, \\ &\vdots \end{aligned}$$

In general, the condition $P(x) = C^n \in K[x]$ yields equations $(C^n)_{-k} = 0$, whereas the condition $Q(x) = \sum_{i=0}^{m+n-2} \lambda_i C^{m-i} + F \in K[x]$ gives us the equations $\left(\sum_{i=0}^{m+n-2} \lambda_i C^{m-i} + F\right)_{-k} = 0$, where we note that $F_{-k} = 0$ for $k = 1, \dots, n-2$.

It is easy to see (e.g. [2, Remark 1.13]) that the first $m+n-2$ coefficients determine the others, i.e., the coefficients $C_{-1}, \dots, C_{-m-n+2}$ determine univocally the coefficients C_{-k} for $k > m+n-2$. Moreover, the F_{-k} for $k > n-1$ depend only on F_{1-n} and C . Consequently, having a solution C to the system $S(n, m, (\lambda_i), F_{1-n})$ is the same as having a solution $(C_{-1}, \dots, C_{-m-n+2})$ to the system

$$\begin{aligned} E_k &:= (C^n)_{-k} = 0, & \text{for } k = 1, \dots, m-1, \\ E_{m-1+k} &:= \left(\sum_{i=0}^{m+n-2} \lambda_i C^{m-i} \right)_{-k} = 0, & \text{for } k = 1, \dots, n-2, \\ E_{m+n-2} &:= \left(\sum_{i=0}^{m+n-2} \lambda_i C^{m-i} \right)_{1-n} + F_{1-n} = 0, \end{aligned} \tag{1.3}$$

with $m+n-2$ equations E_k and $m+n-2$ unknowns C_{-k} .

In order to understand the solution set of this system, it would be very helpful to find a Groebner basis for the ideal generated by the polynomials E_k in $D[C_{-1}, \dots, C_{m+n-2}]$. In this paper we compute such a Groebner basis of (1.3) in a very particular case: we assume $n = 2$, $m = 2r+1$ for some $r > 0$, and $\lambda_i = 0$ for $i > 0$. Moreover we consider $D = \mathbb{C}[y]$ and $F_{1-n} = y$, as in Theorem 1.1.

2 Computation of a Groebner basis for I_{2r}

Assume $n = 2$, $m = 2r+1$ for some $r > 0$, and $\lambda_i = 0$ for $i > 0$. Set also $D = \mathbb{C}[y]$ and $F_{1-n} = y$.

Then the system (1.3) reads

$$E_i = \begin{cases} (C^2)_{-i}, & i = 1, \dots, 2r \\ (C^{2r+1})_{-1} + y, & i = 2r+1, \end{cases} \tag{2.1}$$

where $(C^2)_{-i}$ denotes the coefficient of x^{-i} in the Laurent series C^2 . Explicitly, the polynomials E_i are given by

$$\begin{aligned}
E_1 &:= 2C_{-2}, \\
E_2 &:= 2C_{-3} + C_{-1}^2, \\
E_3 &:= 2C_{-4} + 2C_{-2}C_{-1}, \\
E_4 &:= 2C_{-5} + 2C_{-3}C_{-1} + C_{-2}^2, \\
E_5 &:= 2C_{-6} + 2C_{-2}C_{-3} + 2C_{-4}C_{-1}, \\
E_6 &:= 2C_{-7} + 2C_{-5}C_{-1} + 2C_{-4}C_{-2} + C_{-3}^2, \\
&\vdots \\
E_{2r-1} &:= 2C_{-2r} + 2C_{-2}C_{-2r+3} + 2C_{-4}C_{-2r+5} + \cdots + 2C_{-2r+4}C_{-3} + 2C_{-2r+2}C_{-1}, \\
E_{2r} &:= 2C_{-2r-1} + 2C_{-2r+1}C_{-1} + 2C_{-2r+2}C_{-2} + \cdots + C_{-r}^2, \\
E_{2r+1} &:= (C^{2r+1})_{-1} + y.
\end{aligned} \tag{2.2}$$

Each E_i is a polynomial in the ring $\mathbb{C}[C_{-1}, C_{-2}, \dots, C_{-2r-1}, y]$, and the $2r+1$ polynomials yield the ideal

$$I = \langle E_1, \dots, E_{2r}, E_{2r+1} \rangle.$$

Our goal is to find a Groebner basis for the ideal I . However, in this section we will only compute a Groebner basis $(\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{2r-1}, \tilde{E}_{2r})$ for the ideal $I_{2r} := \langle E_1, E_2, \dots, E_{2r-1}, E_{2r} \rangle$. Note that for $i = 1 \dots, 2r$ we have

$$E_i = 2C_{-i-1} + \sum_{k=1}^{i-1} C_{-k}C_{k-i}. \tag{2.3}$$

We first replace the odd numbered polynomials $E_1, E_3, E_5, E_7, \dots, E_{2r-1}$ by new polynomials $\tilde{E}_1, \tilde{E}_3, \tilde{E}_5, \tilde{E}_7, \dots, \tilde{E}_{2r-1}$ defined by

$$\begin{aligned}
\tilde{E}_1 &:= C_{-2} = \frac{1}{2}E_1, \\
\tilde{E}_3 &:= C_{-4} = \frac{1}{2}E_3 - \tilde{E}_1C_{-1}, \\
\tilde{E}_5 &:= C_{-6} = \frac{1}{2}E_5 - \tilde{E}_1C_{-3} - \tilde{E}_3C_{-1}, \\
\tilde{E}_7 &:= C_{-8} = \frac{1}{2}E_7 - \tilde{E}_1C_{-5} - \tilde{E}_3C_{-3} - \tilde{E}_5C_{-1}, \\
\tilde{E}_9 &:= C_{-10} = \frac{1}{2}E_9 - \tilde{E}_1C_{-7} - \tilde{E}_3C_{-5} - \tilde{E}_5C_{-3} - \tilde{E}_7C_{-1}, \\
&\vdots \\
\tilde{E}_{2r-1} &:= C_{-2r} = \frac{1}{2}E_{2r-1} - \sum_{i=1}^{r-1} \tilde{E}_{2i-1}C_{-2(r-i)+1}.
\end{aligned} \tag{2.4}$$

Remark 2.1. *We have*

$$\langle E_1, E_3, \dots, E_{2r-1} \rangle = \langle \tilde{E}_1, \tilde{E}_3, \dots, \tilde{E}_{2r-1} \rangle. \tag{2.5}$$

In fact, if we define $\tilde{I}_k^{\text{odd}} := \langle \tilde{E}_1, \tilde{E}_3, \dots, \tilde{E}_{2k-1} \rangle$, then (2.4) clearly implies

$$E_{2i+1} - 2\tilde{E}_{2i+1} \in \tilde{I}_i^{\text{odd}}, \tag{2.6}$$

and so we get $\langle E_1, E_3, \dots, E_{2i+1} \rangle \subset \langle \tilde{E}_1, \tilde{E}_3, \dots, \tilde{E}_{2i+1} \rangle$ for all i . Using induction one sees that we also have $\langle \tilde{E}_1, \tilde{E}_3, \dots, \tilde{E}_{2r-1} \rangle \subset \langle E_1, E_3, \dots, E_{2r-1} \rangle$, as desired.

The next proposition deals with $E_2, E_4, E_6, \dots, E_{2r}$, the first r even numbered polynomials.

Proposition 2.2. *For all $j \in \mathbb{N}$ there exists λ_j such that for $\tilde{E}_{2j} := C_{-2j-1} + \lambda_j C_{-1}^{j+1}$ we have*

$$C_{-2j-1} + \lambda_j C_{-1}^{j+1} - \frac{1}{2} E_{2j} \in \tilde{I}_{2j-1} := \langle \tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{2j-2}, \tilde{E}_{2j-1} \rangle. \quad (2.7)$$

Moreover, if we set $\lambda_0 = -1$ and $E_0 = \tilde{E}_0 := 0$, then for $j > 0$, λ_j is given by

$$\lambda_j := \frac{1}{2} \left(\sum_{k=0}^{j-1} \lambda_k \lambda_{j-k-1} \right). \quad (2.8)$$

Proof. We proceed by induction on j . For $j = 0$ clearly (2.7) is satisfied. For $j = 1$, with $\lambda_1 = \frac{1}{2}$ calculated by (2.8), we have

$$C_{-3} + \frac{1}{2} C_{-1}^2 - \frac{1}{2} E_2 = 0 \in \langle \tilde{E}_1 \rangle,$$

as desired.

From (2.3) we have

$$\begin{aligned} E_{2j} &= 2C_{-2j-1} + \sum_{k=1}^{2j-1} C_{-k} C_{k-2j} \\ &= 2C_{-2j-1} + \sum_{k=0}^{j-1} C_{-2k-1} C_{2k+1-2j} + \sum_{k=1}^{j-1} C_{-2k} C_{2k-2j}, \end{aligned}$$

and clearly $\sum_{k=1}^{j-1} C_{-2k} C_{2k-2j} \in \tilde{I}_{2j-1}$. Therefore we get

$$C_{-2j-1} - \frac{1}{2} E_{2j} \in -\frac{1}{2} \left(\sum_{k=0}^{j-1} C_{-2k-1} C_{2k+1-2j} \right) + \tilde{I}_{2j-1}. \quad (2.9)$$

By the induction hypothesis, for $0 \leq k \leq j-1$ there exist λ_k and λ_{j-k-1} such that

$$C_{-2k-1} = -\lambda_k C_{-1}^{k+1} + \tilde{E}_{2k} \quad \text{and} \quad C_{2k+1-2j} = -\lambda_{j-k-1} C_{-1}^{j-k} + \tilde{E}_{2(j-k-1)};$$

hence

$$C_{-2k-1} C_{2k+1-2j} \in \lambda_k \lambda_{j-k-1} C_{-1}^{j+1} + \tilde{I}_{2j-1}.$$

From (2.9) we obtain

$$C_{-2j-1} - \frac{1}{2} E_{2j} \in -\frac{1}{2} \left(\sum_{k=0}^{j-1} \lambda_k \lambda_{j-k-1} \right) C_{-1}^{j+1} + \tilde{I}_{2j-1},$$

from which (2.7) follows with $\lambda_j = \frac{1}{2} \left(\sum_{k=0}^{j-1} \lambda_k \lambda_{j-k-1} \right)$, as desired. \square

Corollary 2.3. *We have*

$$\langle E_1, E_2, \dots, E_{2r} \rangle = \langle \tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{2r} \rangle. \quad (2.10)$$

Proof. In fact, if we define $\tilde{I}_k := \langle \tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_k \rangle$, then (2.6) and Proposition 2.2 clearly imply

$$E_{k+1} - 2\tilde{E}_{k+1} \in \tilde{I}_k,$$

and so we get $\langle E_1, E_2, \dots, E_{k+1} \rangle \subset \langle \tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{k+1} \rangle$ for all k . Since $\langle E_1 \rangle = \langle \tilde{E}_1 \rangle$, using induction one also obtains $\langle \tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_k \rangle \subset \langle E_1, E_2, \dots, E_k \rangle$, as desired. \square

We can replace the system (2.2) with the following set of equations.

$$\begin{aligned}\tilde{E}_1 &= C_{-2} = 0, & \tilde{E}_3 &= C_{-4} = 0, & \dots & \tilde{E}_{2r-1} = C_{-2r} = 0, \\ \tilde{E}_2 &= C_{-3} + \lambda_1 C_{-1}^2 = 0, & \tilde{E}_4 &= C_{-5} + \lambda_2 C_{-1}^3 = 0, & \dots & \tilde{E}_{2r} = C_{-2r-1} + \lambda_r C_{-1}^{r+1} = 0, \\ E_{2r+1} &= (C^{2r+1})_{-1} + y = 0.\end{aligned}$$

Proposition 2.4. *If we fix the lex order with $C_{-2r-1} > C_{-2r} > \dots > C_{-3} > C_{-2} > C_{-1} > y$, then $G_{2r} = (\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{2r-1}, \tilde{E}_{2r})$ is a Groebner basis of the ideal*

$$\tilde{I}_{2r} = \langle \tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{2r-1}, \tilde{E}_{2r} \rangle$$

Proof. We first compute the S -polynomials of G_{2r} , and prove that they satisfy $\overline{S(\tilde{E}_i, \tilde{E}_j)}^{G_{2r}} = 0$ for all $1 \leq i, j \leq 2r$.

Consider first the S -polynomial of an even-numbered polynomial and an odd numbered polynomial. So look now at \tilde{E}_{2s-1} and \tilde{E}_{2t} , with $1 \leq s, t \leq r$. We have

$$\begin{aligned}S(\tilde{E}_{2s-1}, \tilde{E}_{2t}) &= C_{-2t-1}C_{-2s} - C_{-2s}(C_{-2t-1} + \lambda_t C_{-1}^{t+1}) \\ &= -\lambda_t C_{-1}^{t+1} C_{-2s} \\ &= -\lambda_t C_{-1}^{t+1} \tilde{E}_{2s-1},\end{aligned}$$

and so $\overline{S(\tilde{E}_{2s-1}, \tilde{E}_{2t})}^{G_{2r}} = 0$, for all $1 \leq s, t \leq r$.

In the case that i, j are both odd, we take $\tilde{E}_{2s-1}, \tilde{E}_{2t-1}$, with $1 \leq s, t \leq r$. Then we have

$$S(\tilde{E}_{2s-1}, \tilde{E}_{2t-1}) = C_{-2t}C_{-2s} - C_{-2s}C_{-2t} = 0,$$

and trivially we get $\overline{S(\tilde{E}_{2s-1}, \tilde{E}_{2t-1})}^{G_{2r}} = 0$, for all $1 \leq s, t \leq r$.

In the last case, when i, j are even, consider $\tilde{E}_{2s}, \tilde{E}_{2t}$, with $1 \leq s, t \leq r$. Then we have

$$\begin{aligned}S(\tilde{E}_{2s}, \tilde{E}_{2t}) &= C_{-2t-1}(C_{-2s-1} + \lambda_s C_{-1}^{s+1}) - C_{-2s-1}(C_{-2t-1} + \lambda_t C_{-1}^{t+1}) \\ &= \lambda_s C_{-1}^{s+1} C_{-2t-1} - \lambda_t C_{-1}^{t+1} C_{-2s-1}.\end{aligned}$$

Now we divide $S(\tilde{E}_{2s}, \tilde{E}_{2t})$ by G_{2r} . If $C_{-2t-1} > C_{-2s-1}$, then the leading term is

$$lt(S(\tilde{E}_{2s}, \tilde{E}_{2t})) = \lambda_s C_{-1}^{s+1} C_{-2t-1}$$

and the first division step yields

$$S(\tilde{E}_{2s}, \tilde{E}_{2t}) = \lambda_s C_{-1}^{s+1} \tilde{E}_{2t} + R_1,$$

with $R_1 = -\lambda_s \lambda_t C_{-1}^{s+t+2} - \lambda_t C_{-1}^{t+1} C_{-2s-1}$. But continuing the division algorithm we obtain

$$R_1 = -\lambda_t C_{-1}^{t+1} \tilde{E}_{2s},$$

and hence $\overline{S(\tilde{E}_{2s}, \tilde{E}_{2t})}^{G_{2r}} = 0$ in this case. The case $C_{-2s-1} > C_{-2t-1}$ is similar, so we get $\overline{S(\tilde{E}_{2t}, \tilde{E}_{2s})}^{G_{2r}} = 0$ for all $1 \leq s, t \leq r$. \square

From Corollary 2.3 and Proposition 2.4 we conclude that $(\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{2r-1}, \tilde{E}_{2r})$ is a Groebner basis for the ideal $\langle E_1, E_2, \dots, E_{2r-1}, E_{2r} \rangle$.

3 A recursive formula for the Catalan numbers and a Groebner basis

In this last section we will determine a Groebner basis for the ideal I given by the complete system (2.1). In order to achieve this, we need to establish additional properties of the λ_j 's which are closely related to the ubiquitous Catalan numbers.

Lemma 3.1. *For all $j \geq 0$ the equality*

$$c_j = (-1)^{j+1} 2^j \lambda_j \quad (3.1)$$

holds, where c_j are the Catalan numbers given by

$$c_j = \frac{1}{j+1} \binom{2j}{j}.$$

Proof. The Catalan numbers are uniquely determined (see e.g. [3, p.117 (5.6)]) by $c_0 = 1$ and the recursive relation

$$c_r = \sum_{j=0}^{r-1} c_j c_{r-1-j}.$$

Set $d_j = (-1)^{j+1} 2^j \lambda_j$. Then $d_0 = 1$, since $\lambda_0 = -1$, and equality (2.8) gives us

$$\begin{aligned} d_j &= (-1)^{j+1} 2^j \lambda_j \\ &= (-1)^{j+1} 2^j \frac{1}{2} \left(\sum_{k=0}^{j-1} \lambda_k \lambda_{j-k-1} \right) \\ &= \sum_{k=0}^{j-1} ((-1)^{k+1} 2^k \lambda_k) ((-1)^{j-k} 2^{j-1-k} \lambda_{j-k-1}) \\ &= \sum_{k=0}^{j-1} d_k d_{j-1-k}, \end{aligned}$$

and hence $d_j = c_j$ for all j , as desired. □

Now we prove a recursive formula for the Catalan numbers.

Proposition 3.2. *The Catalan numbers satisfy the following formula*

$$(2r+1) \frac{c_r}{2^{2r}} = \sum_{j=0}^r (-1)^j \binom{r}{j} \frac{c_j}{2^{2j}}. \quad (3.2)$$

Consequently, λ_r satisfies

$$(2r+1)(-1)^{r+1} \lambda_r = \sum_{j=0}^r \binom{r}{j} 2^{r-j} (-\lambda_j). \quad (3.3)$$

Proof. Replacing c_j in (3.2) and using (3.1) yields (3.3), hence, it suffices to prove only (3.2). For that, we replace c_j by $\frac{1}{j+1} \binom{2j}{j}$ on the righthand side of (3.2) and use the equalities

$$\binom{-1/2}{j} = \frac{(-1)^j}{2^{2j}} \binom{2j}{j} \quad \text{and} \quad \binom{r+1/2}{r} = \frac{(2r+1)}{2^{2r}} \binom{2r}{r}.$$

Then we have

$$\begin{aligned}
\sum_{j=0}^r (-1)^j \binom{r}{j} \frac{c_j}{2^{2j}} &= \sum_{j=0}^r \frac{(-1)^j}{2^{2j}} \binom{2j}{j} \cdot \frac{1}{(j+1)} \binom{r}{j} \\
&= \sum_{j=0}^r \binom{-1/2}{j} \frac{1}{r+1} \binom{r+1}{j+1} \\
&= \frac{1}{(r+1)} \sum_{j=0}^r \binom{-1/2}{j} \cdot \binom{r+1}{r-j} \\
&= \frac{1}{(r+1)} \binom{r+1/2}{r} \\
&= \frac{1}{(r+1)} \frac{(2r+1)}{2^{2r}} \binom{2r}{r} \\
&= (2r+1) \frac{c_r}{2^{2r}}.
\end{aligned}$$

The second equality follows from the relation $\frac{1}{j+1} \binom{r}{j} = \frac{1}{(r+1)} \binom{r+1}{j+1}$ and the fourth equality from $\binom{\alpha+\beta}{r} = \sum_{j=0}^r \binom{\alpha}{j} \binom{\beta}{r-j}$, valid for all $\alpha, \beta \in \mathbb{Q}$. \square

Proposition 3.3. *Let $I_{2r} = \langle E_1, E_2, \dots, E_{2r} \rangle$. Then*

$$(C^{2r+1})_{-1} \in \mu_r C_{-1}^{r+1} + I_{2r},$$

for $\mu_r = \frac{2r+1}{(r+1)2^r} \binom{2r}{r}$.

Proof. By definition we have

$$(C^{2r+1})_{-1} = [(C^2)^r C]_{-1} = \sum_{j=-2}^{2r} [(C^2)^r]_j C_{-j-1},$$

since $C_{-j-1} = 0$ for $j < -2$ and $[(C^2)^r]_j = 0$ for $j > 2r$.

But we also have $[(C^2)^r]_j = \sum_{i_1+\dots+i_r=j} (C^2)_{i_1} \dots (C^2)_{i_r}$. We claim that $i_k \geq -2r$. In fact, as $i_j \leq 2$, then so we have

$$i_1 + \dots + i_{k-1} + i_{k+1} + \dots + i_r \leq 2(r-1),$$

and $j = i_k + (i_1 + \dots + i_{k-1} + i_{k+1} + \dots + i_r) \leq 2(r-1) + i_k$ as well. Therefore we get $i_k \geq j - 2r + 2 \geq -2r$, since $j \geq -2$.

By definition we have $E_i = (C^2)_{-i}$ for $i = 1, \dots, 2r$. Consequently we obtain

$$(C^2)_{i_1} \dots (C^2)_{i_r} \in I_{2r}, \quad \text{if some } i_k < 0.$$

It follows that

$$[(C^2)^r]_j \in \sum_{\substack{i_1+\dots+i_r=j \\ i_k \geq 0}} (C^2)_{i_1} \dots (C^2)_{i_r} + I_{2r} = [(x^2 + 2C_{-1})^r]_j + I_{2r}, \quad (3.4)$$

holds, since $C^2 = x^2 + 2C_{-1} + (C^2)_{-1}x^{-1} + (C^2)_{-2}x^{-2} + (C^2)_{-3}x^{-3} + \dots$. But we also have

$$(x^2 + 2C_{-1})^r = \sum_{k=0}^r \binom{r}{k} (2C_{-1})^{r-k} x^{2k},$$

and so

$$[(x^2 + 2C_{-1})^r]_j = \begin{cases} \binom{r}{k}(2C_{-1})^{r-k} & \text{if } j = 2k \\ 0, & \text{if } j = 2k + 1. \end{cases}$$

We arrive at

$$(C^{2r+1})_{-1} \in \sum_{k=0}^r \binom{r}{k} (2C_{-1})^{r-k} C_{-2k-1} + I_{2r}.$$

Note that by Proposition 2.2 we have

$$C_{-2k-1} = \tilde{E}_{2k} - \lambda_k C_{-1}^{k+1} \in -\lambda_k C_{-1}^{k+1} + I_{2r},$$

so we obtain

$$\begin{aligned} (C^{2r+1})_{-1} &\in \sum_{k=0}^r \binom{r}{k} (2C_{-1})^{r-k} (-\lambda_k C_{-1}^{k+1}) + I_{2r} \\ &= \left(\sum_{k=0}^r \binom{r}{k} 2^{r-k} (-\lambda_k) \right) (C_{-1})^{r+1} + I_{2r}, \end{aligned}$$

and the formula for μ_r follows now from (3.1) and (3.3). \square

Corollary 3.4. *For $\tilde{E}_{2r+1} := \mu_r(C_{-1})^{r+1} + y$ we have*

$$\langle E_1, E_2, \dots, E_{2r-1}, E_{2r}, E_{2r+1} \rangle = \langle \tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{2r-1}, \tilde{E}_{2r}, \tilde{E}_{2r+1} \rangle.$$

Proof. By Proposition 3.3 we have $E_{2r+1} - \tilde{E}_{2r+1} = (C^{2r+1})_{-1} - \mu_r C_{-1}^{r+1} \in I_{2r}$, hence the corollary follows from Corollary 2.3. \square

Now we can state our main result.

Theorem 3.5. *If we fix the lex order with $C_{-2r-1} > C_{-2r} > \dots > C_{-3} > C_{-2} > C_{-1} > y$, then $G_{2r+1} = (\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{2r}, \tilde{E}_{2r+1})$ is a Groebner basis for the ideal*

$$I = \langle E_1, E_2, \dots, E_{2r-1}, E_{2r}, E_{2r+1} \rangle.$$

Proof. By Corollary 3.4 it suffices to prove that the division of the S -polynomials $S(\tilde{E}_i, \tilde{E}_j)$ by G_{2r+1} is zero. If $i, j \leq 2r$, then the division algorithm yields the same quotients and remainders as in Proposition 2.4, since the remainders become zero before one has to divide by \tilde{E}_{2r+1} . Note that $lt(\tilde{E}_{2r+1}) = \mu_r(C_{-1})^{r+1}$, since $\mu_r \neq 0$. It remains to divide the S -polynomials $S(\tilde{E}_i, \tilde{E}_{2r+1})$ by G_{2r+1} . We first consider the case $i = 2t - 1$ for some $t = 1, \dots, r$. We get

$$\begin{aligned} S(\tilde{E}_{2t-1}, \tilde{E}_{2r+1}) &= \frac{C_{-2t}C_{-1}^{r+1}}{C_{-2t}}(C_{-2t}) - \frac{C_{-2t}C_{-1}^{r+1}}{\mu_r C_{-1}^{r+1}}(\mu_r C_{-1}^{r+1} + y) \\ &= -\frac{1}{\mu_r} y C_{-2t}, \end{aligned}$$

for all $t = 1, \dots, r$. The first division step yields $S(\tilde{E}_{2t-1}, \tilde{E}_{2r+1}) = -\frac{1}{\mu_r} \tilde{E}_{2t-1}$, hence we obtain $\frac{S(\tilde{E}_{2t-1}, \tilde{E}_{2r+1})}{S(\tilde{E}_{2t-1}, \tilde{E}_{2r+1})}^{G_{2r+1}} = 0$, for all $t = 1, \dots, r$.

Now for the S -polynomials of \tilde{E}_{2t} and \tilde{E}_{2r+1} , for some $t = 1, \dots, r$, we have

$$\begin{aligned} S(\tilde{E}_{2t}, \tilde{E}_{2r+1}) &= \frac{C_{-2t-1}C_{-1}^{r+1}}{C_{-2t-1}}(C_{-2t-1} + \lambda_t C_{-1}^{t+1}) - \frac{C_{-2t-1}C_{-1}^{r+1}}{\mu_r C_{-1}^{r+1}}(\mu_r C_{-1}^{r+1} + y) \\ &= \lambda_t C_{-1}^{r+t+2} - \frac{1}{\mu_r} C_{-2t-1} y. \end{aligned}$$

with leading term

$$lt(S(\tilde{E}_{2t}, \tilde{E}_{2r+1})) = -\frac{1}{\mu_r} C_{-2t-1} y.$$

We divide $S(\tilde{E}_{2t}, \tilde{E}_{2r+1})$ by G_{2r+1} , and the first division step gives us

$$S(\tilde{E}_{2t}, \tilde{E}_{2r+1}) = -\frac{1}{\mu_r} y \tilde{E}_{2t} + R_1$$

with $R_1 = \lambda_t C_{-1}^{r+t+2} + \frac{\lambda_t}{\mu_r} y C_{-1}^{t+1}$. Finally we note that $R_1 = \frac{\lambda_t}{\mu_r} C_{-1}^{t+1} \tilde{E}_{2r+1}$, in order to obtain $\frac{S(\tilde{E}_{2t}, \tilde{E}_{2r+1})}{G_{2r+1}} = 0$, for all $t = 1, \dots, r$. This concludes the proof. \square

We give explicitly the Groebner basis $G_{2r+1} = (\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{2r}, \tilde{E}_{2r+1})$ of I as

$$\begin{aligned} \tilde{E}_1 &= C_{-2}, & \tilde{E}_3 &= C_{-4}, & \dots & \tilde{E}_{2r-1} &= C_{-2r} \\ \tilde{E}_2 &= C_{-3} + \lambda_1 C_{-1}^2, & \tilde{E}_4 &= C_{-5} + \lambda_2 C_{-1}^3, & \dots & \tilde{E}_{2r} &= C_{-2r-1} + \lambda_r C_{-1}^{r+1} \\ \tilde{E}_{2r+1} &= \mu_r (C_{-1})^{r+1} + y, \end{aligned}$$

with

$$\mu_r = \frac{2r+1}{(r+1)2^r} \binom{2r}{r} \quad \text{and} \quad \lambda_j = \frac{(-1)^{j+1}}{(j+1)2^j} \binom{2j}{j}.$$

References

- [1] David Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms*, 3rd ed., Undergraduate Texts in Mathematics, Springer, New York, 2007. An introduction to computational algebraic geometry and commutative algebra.
- [2] Jorge Alberto Guccione, Juan José Guccione, and Christian Valqui, *A system of polynomial equations related to the Jacobian conjecture*, arXiv:1406.0886v1 [math.AG] (3 June 2014).
- [3] Thomas Koshy, *Catalan numbers with applications*, Oxford University Press, Oxford, 2009.

CHRISTIAN VALQUI, PONTIFICIA UNIVERSIDAD CATÓLICA DEL PERÚ, SECCIÓN MATEMÁTICAS, PUCP, AV. UNIVERSITARIA 1801, SAN MIGUEL, LIMA 32, PERÚ.

INSTITUTO DE MATEMÁTICA Y CIENCIAS AFINES (IMCA) CALLE LOS BIÓLOGOS 245. URB SAN CÉSAR. LA MOLINA, LIMA 12, PERÚ.

E-mail address: cvalqui@pucp.edu.pe

MARCO SOLORZANO, PONTIFICIA UNIVERSIDAD CATÓLICA DEL PERÚ, SECCIÓN MATEMÁTICAS, PUCP, AV. UNIVERSITARIA 1801, SAN MIGUEL, LIMA 32, PERÚ.

E-mail address: marco.solorzano@pucp.pe